Convective instability of magnetic fluids

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A theoretical investigation of the convective instability problem in the thin horizontal layer of a magnetic fluid heated from below is carried out. The effects of the relaxation time τ and the vortex (rotational) viscosity ξ are considered and discussed. The Chebyshev pseudospectral method is employed to solve the eigenvalue problems and numerical calculations are carried out for a number of magnetic fluids and in full range of the magnetic field. A variety of results under gravity-free conditions are also presented and the critical temperature gradient are determined for a variety of situations. It is shown that the consideration of (ξ,τ), in the stability analysis, is most effective in the thin layer of the fluid and at low values of Langevin parameter α_L.

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I. INTRODUCTION

Magnetic fluids or ferrofluids are colloidal suspensions of fine magnetic mono domain nanoparticles in nonconducting liquids like hydrocarbon, ester, water, etc. These fluids are not found in nature but are artificially synthesized. The continuum description of the magnetic fluids, termed as ferrohydrodynamics, has been in existence since the work of Neuringer and Rosensweig [1]. This so-called quasistationary theory has been helpful to explain many physical phenomena and has a wide range of applications [2,3]. The theory has, however, limitations because it assumes that magnetization is always collinear to the applied field at every instant, that is, magnetization relaxation time is zero. Recent researchers have shown that when magnetic fluids move in the presence of an applied field, or are subjected to unsteady magnetic fields, the phenomena observed can only be explained by a finite magnetization relaxation time [2,4–7]. There are, in general, two mechanisms of magnetization relaxation. The first one is Néel mechanism in which the particle magnetic moment **m** rotates inside the particle and the particle does not rotate itself. In Brownian relaxation mechanism the vector **m** is locked into the crystal axis of the particle and rotates along with the particle rotation. Thus when the applied field has a changing direction or magnitude, in the Néel mechanism, the relaxation of the magnetization does not initiate the rotation of the particle and accordingly no momentum transfer, from the particle to the fluid, occurs. In Brownian mechanism the magnetization relaxation, however, causes the rotation of the particle itself and there is thus momentum transfer to the carrier fluid in the form of a viscous friction. It is believed that quasistationary theory [1] is reasonably valid for colloidal suspensions of Néel particles. For Brownian particle, with finite magnetic relaxation time, one, however, needs to incorporate the intrinsic rotation of the particle in the theory [2,4,5].

There have been a number of convective instability studies using the quasistationary theory [1]. Finlayson [8] studied the convective instability of a magnetic fluid for a fluid layer heated from below in the presence of a uniform vertical magnetic field. He discussed both shear free and rigid horizontal boundaries using the linear stability method. Gotoh and Yamada [9] carried out the same study by assuming the fluid to be confined between two magnetic pole pieces. Lalas and Carmi [10] analyzed the problem, studied in Ref. [8], by energy stability method. Convective instability analysis for a rotating layer of a magnetic fluid between two free boundaries and between rigid or ferromagnetic boundaries have been studied by Gupta and Gupta [11], and Venkatasubramanian and Kaloni [12], respectively. Qin and Kaloni [13] have analyzed the buoyancy-surface tension effects, using linear and nonlinear theory, in a magnetic fluid heated from below. Recently Shivakumara *et al.* [14] have considered the nonuniform basic temperature gradient on Rayleigh-Benard-Marangoni convection. Some other aspects of the theromoconvection problems, in magnetic fluids, have also been studied by Recktenwald and Lücke [15], Blennerhassett *et al.* [16], and Straughan [17].

In two recent papers Shliomis [18] and Shliomis and Smorodin [19] have studied the convective instability of magnetized ferrofluids by treating the fluids as binary mixtures. These authors consider the influence of concentration gradients due to magnetophoresis and Soret effects. As Shliomis [18] points out this situation is possible only when the temperature difference is allowed to increase very slowly so that the mass diffusion develops and remains undisturbed before the convection starts due to temperature difference. Unlike the results of previous authors [9–16], who found only stationary instability to occur in all cases, Shliomis and Smorodin [19] predict oscillatory instability in a certain region of magnetic field and the fluid temperature. Equations used in Refs. [18,19] are again those of quasistationary theory [1].

Stiles and Kagan [20] discuss the thermoconvective instability of a ferrofluid in a strong external magnetic field. These authors consider the equations involving rotational or vortex viscosity and the nonequilibrium magnetization equation, involving Brownian relaxation time τ. Unfortunately, because of the strong external field assumption, which soon leads to magnetic saturation, and in the manner equations are linearized and approximated, their equations determining the stability criteria turn out to be identical to those of Finlayson [8] [compare Eqs. (4.11) – (4.15) of Ref. [20] with Eq. (17) (18a,b,c) of Ref. [8]]. In particular, these authors assume τ tending to zero and also the average spin of the particle equal to the fluid vorticity. Stiles and Kagan [20], however, con-

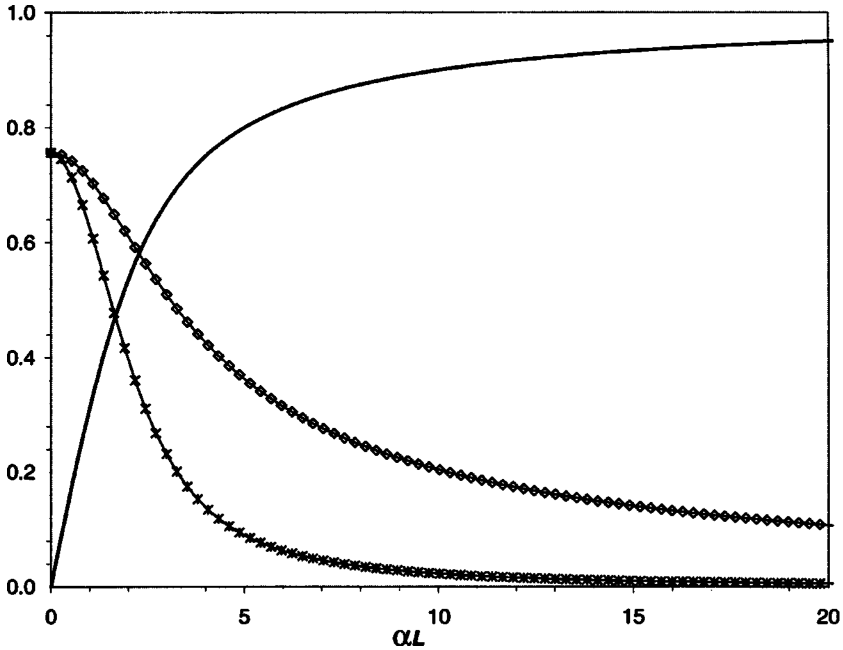

FIG. 1. Plots of $L(\alpha_L)$ (solid line), χ (line with *), χ_2 (line with $\diamondsuit$) against α_L, for a ferrofluid with $M_s = 31\,800$ A/m, $T_a = 298$ K.

sider the effect of the dependence of shear viscosity on colloidal concentration and on temperature gradient, not studied in Ref. [8]. We remark that, in a proper dilute theory of a magnetic fluid accounting for internal rotation of the particles, the influence of the relaxation time τ and the vortex (rotational) viscosity ξ, in the absence of colloidal particle inertia which is usually very small, are tied together. Thus the absence of one variable in a physical situation implies, automatically, the absence of the other variable and vice versa.

In the present paper we consider the convective instability problem in the horizontal layer of a magnetic fluid with Brownian relaxation mechanism. We employ appropriate equations which allow proper consideration of internal rotation and vortex viscosity [2,4,5], and present a reasonably complete study of the problem. In order to gain some feeling about the values of various basic parameters involved, we have sketched them in Figs. 1 and 2. We use Chebyshev pseudospectral method to solve the eigenvalue problems. It turns out that, apart from the Rayleigh number Rg and magnetic Rayleigh number N, several other parameters involving τ and ξ dictate the stability pattern. We consider the ferrofluid layers varying from 1 mm to 8 mm and carry out calculations for three carrier fluids, hydrocarbon, ester and water. Our results, thus, not only predict the effect of τ and ξ consideration on the stability, but we also report the magnetic field effect in its entire range (from low values to high values leading to magnetic saturation). In this manner, in the absence of (ξ, τ) consideration, we complement the results of Stiles and Kagan [20] for full range of values of the magnetic field. We also report the critical temperature values to start the convection.

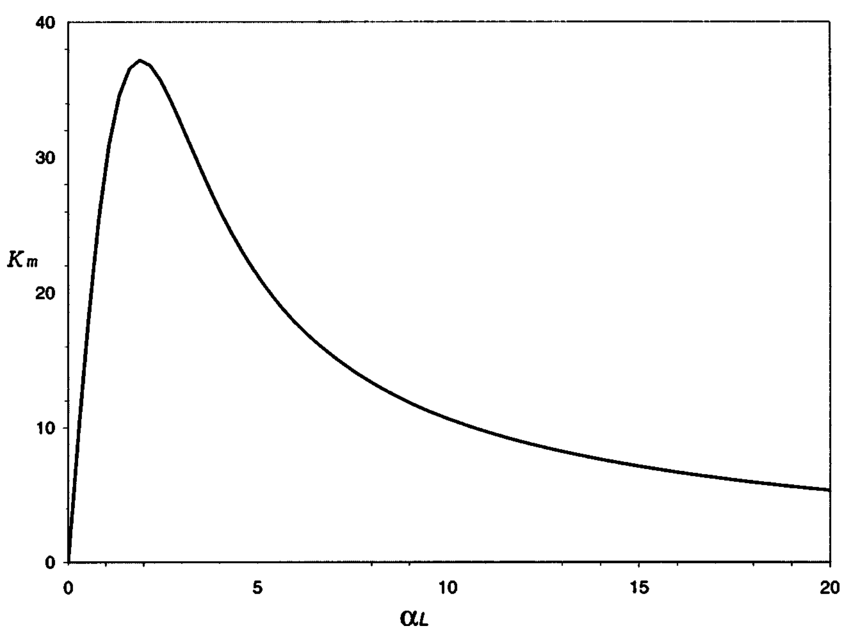

FIG. 2. Variation of K/m with α_L for a ferrofluid with $M_s = 31\,800$ A/m, $T_a = 298$ K.

II. BASIC EQUATIONS

The equations governing the flow of an incompressible magnetic fluid, neglecting the spin diffusion terms are given as [2,4,5]

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P' + (\eta + \xi)\nabla^2\mathbf{u} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + 2\xi(\nabla \times \boldsymbol{\omega}) - \rho g\mathbf{k}, \tag{2}$$

$$\rho_0 I \frac{D\omega}{Dt} = 2\xi(\nabla \times \mathbf{u} - 2\boldsymbol{\omega}) + \mu_0 \mathbf{M}\times\mathbf{H} \tag{3}$$

$$\frac{D\mathbf{M}}{Dt} = \boldsymbol{\omega}\times\mathbf{M} - \frac{1}{\tau_m}(\mathbf{M} - \mathbf{M}_{\text{eq}}), \tag{4}$$

and

$$\left[\rho C_{V,H} - \mu_0 \mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial T}\right]\frac{DT}{Dt} + \mu_0 T\left(\frac{\partial \mathbf{M}}{\partial T}\right)_{V,H}\cdot\frac{D\mathbf{H}}{Dt} = K_t\nabla^2 T + \Phi. \tag{5}$$

These are the equations of mass balance, linear momentum, angular momentum, magnetization relaxation, and temperature equations, respectively. Here $\mathbf{u}=(u,v,w)$ is the velocity, $D/Dt=\partial/\partial t+\mathbf{u}\cdot\nabla\rho$ the density, $P'=P+\frac{1}{2}u_0H^2$, $\boldsymbol{\omega}$ is the average spin velocity of colloidal particles, η is the viscosity of carrier fluid, ξ is the vortex (rotational) viscosity, $\mathbf{H}$ is the magnetic field, $\mathbf{M}$ is the magnetization, $\mathbf{M}_{\text{eq}}$ is equilibrium magnetization, μ_0 is magnetic permeability (in free space $\mu_0=4\pi\times10^{-7}$ H/m), ρI is the average moment of inertia of the colloidal particles per unit volume, τ_m is the Brownian relaxation time, T is the temperature, $C_{V,H}$ is the specific heat capacity at constant volume and magentic field, K_t is the thermal conductivity, and Φ is the viscous dissipation. We point out that temperature equation (5) can also be written in an alternate form using specific heat capacity at constant pressure. Maxwell's equations in the magnetostatic limit are

$$\nabla\cdot\mathbf{B}=0, \quad \nabla\times\mathbf{H}=\mathbf{0}, \quad \mathbf{B}=\mu_0(\mathbf{M}+\mathbf{H}). \tag{6}$$

On assuming Boussinesq approximation for the density variation we have

$$\rho g = \rho_0 g[1-\alpha(T-T_a)], \tag{7}$$

where α is the thermal expansion coefficient and T_a is average temperature. Moreover, on neglecting the inertia of the colloidal suspended particles, we can write (3) as

$$\boldsymbol{\omega} = \frac{\mu_0}{4\xi}\mathbf{M}\times\mathbf{H} + \frac{1}{2}(\nabla\times\mathbf{u}). \tag{8}$$

On substituting (7) and (8) into (2) and (4) we have

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P' + \eta\nabla^2\mathbf{u} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \frac{1}{2}\mu_0\nabla\times(\mathbf{M}\times\mathbf{H}) - \rho_0 g(1-\alpha(T-T_a))\mathbf{k}, \tag{9}$$

$$\frac{D\mathbf{M}}{Dt} = \frac{1}{2}(\nabla\times\mathbf{u})\times\mathbf{M} - \frac{1}{\tau_m}(\mathbf{M}-\mathbf{M}_{\text{eq}}) - \frac{\mu_0}{4\xi}\mathbf{M}\times(\mathbf{M}\times\mathbf{H}). \tag{10}$$

Equations (1), (5), (6), (9), and (10) are the basic equations in our work.

At equilibrium, magnetization is aligned with stationary magnetic field and is a function of the magnetic field and the temperature. It is assumed to be described by Langevin formula [4],

$$\mathbf{M}_{\text{eq}} = \frac{\mathbf{H}}{H}M_s L(\alpha_L) = \frac{\mathbf{H}}{H}M_{\text{eq}}L(H,T), \tag{11}$$

$$L(\alpha_L) = \coth(\alpha_L) - \frac{1}{\alpha_L}, \quad \alpha_L = \frac{mH}{k_BT},$$

where $\mathbf{m}$ is the magnetic moment of a single particle, M_s is the saturated magnetization, $k_B=1.38\times10^{-23}$ J/K^{-1} is Boltzmann's constant, and α_L is Langevin parameter.

We consider a horizontal layer of an incompressible magnetic fluid heated from below. A Cartesian coordinate system (x,y,z) is used with z-axis normal to the layer which is confined between horizontal plates $z=-\frac{1}{2}d$ and $z=\frac{1}{2}d$. A uniform magnetic field is applied normal to the plates and a constant temperature gradient is maintained between the plates. The temperature boundary conditions, thus, are

$$T=T_0 \quad \text{at } z=\tfrac{1}{2}d, \quad T=T_1 \quad \text{at } z=-\tfrac{1}{2}d, \quad T_a=\tfrac{1}{2}(T_0+T_1). \tag{12}$$

For velocity we employ no slip boundary conditions, $\mathbf{u}=\mathbf{0}$ on the rigid plates. The magnetic boundary conditions are that the tangential component of the magnetic field and normal component of the magnetic induction are continuous across the boundary.

To obtain the solution in the quiescent state we first linearize magnetization equation M_{eq} in the same manner as Finlayson [8] did

$$M_{\text{eq}} = M_a + \chi(H-H_a) - K_m(T-T_a), \tag{13}$$

where $\chi=(\partial M/\partial H)_{H_a,T_a}$ is tangent magnetic susceptibility and $K_m=-(\partial M/\partial T)_{H_a,T_a}$ is called pyromagnetic coefficient. In our case, $H_x \ll H_z, H_y \ll H_z$, and $H_z \approx H_a$, and simplifying Eqs. (11) and (13), yields

$$M_{\text{eq}\ x} = \chi_2 H_x,$$

$$M_{\text{eq}\ y} = \chi_2 H_y,$$

$$M_{\text{eq}\ z} = M_a + \chi(H_z-H_a) - K_m(T-T_a), \tag{14}$$

where $M_a=\chi_2H_a$. We note that the susceptibility χ (tangent) and χ_2 (chord) and $K_m=\chi H_a/T_a$ are, in general, functions of temperature and magnetic field. Figures 1 and 2 display their characteristics. The tangent and chord magnetization susceptibility χ, χ_2 can be estimated by the Langevin formula (11) for a different Langevin parameter α_L as [2]:

$$\alpha_L = \frac{mH_a}{k_B T_a} = \begin{cases} \ll 1, & \chi = \dfrac{M_s m}{3k_B T_a}, \quad \chi_2 = \chi, \\ \simeq 1, & \chi = \dfrac{M_s m}{k_B T_a} L'(\alpha_L), \quad \chi_2 = \dfrac{M_s}{H_a} L(\alpha_L), \\ \gg 1, & \chi = \dfrac{M_s k_B T_a}{m H_a^2}, \quad \chi_2 = \dfrac{M_s}{H_a}\left(1 - \dfrac{1}{\alpha_L}\right). \end{cases} \tag{15}$$

The quiescent state solution of the basic equations (1), (5), (6), (9), and (10) with corresponding rigid boundary conditions is given to be

$$\mathbf{u}^s = 0, \quad \boldsymbol{\omega}^s = 0, \quad T^s = T_a - \beta z, \quad \beta = \frac{T_1 - T_0}{d}, \tag{16}$$

$$\mathbf{H}^s = \mathbf{k}\left(H_a - \frac{\chi H_a \beta z}{T_a(1+\chi)}\right), \quad \mathbf{M}^s = \mathbf{k}\left(M_a + \frac{\chi H_a \beta z}{T_a(1+\chi)}\right),$$
$$H^s + M^s = H_a^{\text{ext}}, \tag{17}$$

$$p^s = -\rho g z - \frac{\mu_0 \beta \chi H_a M_a}{T_a(1+\chi)} z - \frac{1}{2}\rho g \alpha \beta z^2 - \frac{\mu_0 \beta^2 \chi^2 H_a^2}{2T_a^2(1+\chi)^2} z^2. \tag{18}$$

To study the linear stability of the above solution we now perturb the variables appearing in the above equations. On denoting the perturbation variables by dashes, we write

$$[u, v, w, M_x, M_y, M_z, H_x, H_y, H_z, P, \theta]^T$$
$$= [0,0,0,0,0,M_3^s,0,0,H_3^s,P^s,T^s]^T$$
$$+ [u', v', w', M_x', M_y', M_z', H_x', H_y', H_z', P', \theta']^T. \tag{19}$$

On introducing the following dimensionless quantities:

$$\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad t^* = \frac{K_\theta}{d^2} t, \quad \theta^* = \frac{\theta'}{\beta d}, \quad P^* = \frac{d^2}{\eta K_\theta} P',$$
$$\mathbf{u}^* = \frac{d^2}{K_\theta}\mathbf{u}', \quad \mathbf{M}^* = \frac{\mathbf{M}'}{M_a}, \quad \mathbf{H}^* = \frac{\mathbf{H}}{H_a}, \tag{20}$$

where $K_\theta = Kt/\rho C_{V,H}$ and setting $\mathbf{H}' = \boldsymbol{\nabla}\phi$, the relevant equations, after dropping the primes and asterisks take the form

$$\frac{1}{\Pr}\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nabla^2 u - \frac{\chi_2 M_3 \mathrm{Rg}}{2(1+\chi)} M_x - \psi_1 \frac{\partial M_x}{\partial z} - \frac{M_3 \mathrm{Rg}}{2(1+\chi)}\frac{\partial \phi}{\partial x}$$
$$+ \psi_2 \frac{\partial^2 \phi}{\partial z\, \partial x}, \tag{21}$$

$$\frac{1}{\Pr}\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \nabla^2 v - \frac{\chi_2 M_3 \mathrm{Rg}}{2(1+\chi)} M_y - \psi_1 \frac{\partial M_y}{\partial z} - \frac{M_3 \mathrm{Rg}}{2(1+\chi)}\frac{\partial \phi}{\partial y}$$
$$+ \psi_2 \frac{\partial^2 \phi}{\partial z\, \partial y}, \tag{22}$$

$$\frac{1}{\Pr}\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \nabla^2 w - \frac{\chi_2 M_3 \mathrm{Rg}}{(1+\chi)} M_z - \psi_1 \frac{\partial M_z}{\partial z}$$
$$+ \frac{(1+\chi_2)M_3 M_5}{2}\nabla^2 \phi + \psi_2 \frac{\partial^2 \phi}{\partial z^2} + \mathrm{Rg}\ \theta, \tag{23}$$

$$\frac{\partial M_x}{\partial t} = \left\{\psi_3 - \frac{1}{\tau}\right\} M_x - \left(\frac{1}{2} + \frac{\chi M_4 z}{2\chi_2(1+\chi)}\right)\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$$
$$+ \left\{\psi_4 + \frac{1}{\tau}\right\}\frac{\partial \phi}{\partial x}, \tag{24}$$

$$\frac{\partial M_y}{\partial t} = \left\{\psi_3 - \frac{1}{\tau}\right\} M_y - \left(\frac{1}{2} + \frac{\chi M_4 z}{2\chi_2(1+\chi)}\right)\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$
$$+ \left\{\psi_4 + \frac{1}{\tau}\right\}\frac{\partial \phi}{\partial y}, \tag{25}$$

$$\frac{\partial M_z}{\partial t} = -\frac{1}{\tau} M_z + \frac{\chi}{\chi_2 \tau}\frac{\partial \phi}{\partial z} - \frac{\chi M_4}{\chi_2(1+\chi)} w - \frac{\chi M_4}{\chi_2 \tau}\theta, \tag{26}$$

$$\nabla^2 \phi + \chi_2\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z}\right) = 0, \tag{27}$$

$$\frac{\partial \theta}{\partial t} + \frac{(1+\chi)M_2}{\chi}\left(z - \frac{1}{M_4}\right)\frac{\partial^2 \phi}{\partial t\, \partial z}$$
$$= \nabla^2 \theta + (M_2 M_4 z + 1 - M_2) w, \tag{28}$$

where

$$\psi_1 = \frac{\chi_2 M_3\ \mathrm{Rg}}{2(1+\chi)} z - \frac{\chi_2 M_3\ M_5}{2},$$

$$\psi_2 = \frac{M_3\ \mathrm{Rg}}{2(1+\chi)} z + \frac{\chi_2 M_3\ M_5}{2}, \tag{29}$$

$$\psi_3 = N\left[\frac{z^2}{4\xi_1(1+\chi)} - \frac{(1-\chi_2)z}{4\xi_1 \chi M_4} - \frac{(1+\chi)\chi_2}{4\xi_1 \chi^2 M_4^2}\right],$$

$$\psi_4 = N\left[\frac{z^2}{4\xi_1 \chi_2(1+\chi)} + \frac{z}{2\xi_1 \chi M_4} + \frac{(1+\chi)\chi_2}{4\xi_1 \chi^2 M_4^2}\right],$$

and where the parameters Pr, Rg, M_1, M_2, M_3, M_4, χ, ξ_1, and τ are defined as

$$\Pr = \frac{\eta C_{V,H}}{K_t}, \quad \mathrm{Rg} = \frac{\rho^2 g \alpha \beta C_{V,H} d^4}{\eta K_t}, \quad \tau = \frac{K_t}{\rho C_{V,H} d^2}\tau_m,$$
$$\xi_1 = \frac{\xi}{\eta},$$

$$M_1 = \frac{\mu_0 \beta \chi^2 H_a^2}{\rho g \alpha (1+\chi) T_a^2}, \quad M_2 = \frac{\mu_0 \chi^2 H_a^2}{\rho C_{V,H}(1+\chi) T_a},$$
$$M_3 = \frac{\mu_0 \chi H_a^2}{\rho g \alpha d T_a}.$$

Three related parameters N, M_4, and M_5 are denoted by

$$N = \mathrm{Rg}\ M_1, \quad M_4 = \frac{(1+\chi)M_1}{\chi M_3} = \frac{\beta d}{T_a}, \quad M_5 = \frac{M_3 \mathrm{Rg}}{(1+\chi)M_1} = \frac{\rho^2 g \alpha C_{V,H} T_a d^3}{\chi \eta K_t}.$$

In the above, Pr is the Prandtl number, Rg is the viscous Rayleigh number, and N is the magnetic Rayleigh number. On taking curl curl of Eq. (21)–(23), the vertical component of the resulting equation gives

$$\frac{1}{\mathrm{Pr}}\frac{\partial \nabla^2 w}{\partial t} = \nabla^4 w + \mathrm{Rg}\ \nabla_1^2\theta + \frac{1}{2}M_3M_5\left(\chi_2\frac{\partial \nabla^2 M_z}{\partial z} + \chi_2\nabla_1^2\nabla^2\phi + \nabla^4\phi\right) - \frac{1}{2}M_1M_5\left(\chi_2 z\frac{\partial \nabla^2 M_z}{\partial z} + 2\chi_2\nabla^2 M_z + \frac{\partial^2\nabla^2\phi}{\partial z^2} + 2\frac{\partial \nabla^2\phi}{\partial z}\right), \tag{30}$$

where $\nabla_1^2 = [(\partial^2/\partial x^2) + (\partial^2/\partial y^2)]$.

Taking divergence of magnetization equations (24)–(26), it becomes

$$\frac{\partial \nabla^2\phi}{\partial t} = -\frac{1+\chi_2}{\tau}\nabla_1^2\phi - \frac{1+\chi}{\tau}\frac{\partial^2\phi}{\partial z^2} + \frac{\chi M_4}{\tau}\frac{\partial\phi}{\partial z} + \left(\frac{\chi M_4}{2(1+\chi)}z + \frac{\chi_2}{2}\right)\nabla^2 w + \frac{\chi M_4}{(1+\chi)}\frac{\partial w}{\partial z} + \chi_2\psi_3\frac{\partial M_z}{\partial z} + \psi_3\nabla^2\phi - \chi_2\psi_4\nabla_1^2\phi. \tag{31}$$

Equations (26), (28), (30), and (31) are four required equations for four variables w, M_z, ϕ, and θ. The boundary conditions on velocity and temperature for rigid and rigid plates are

$$w = \frac{\partial w}{\partial z} = \theta = 0, \quad z = \pm\frac{1}{2}. \tag{32}$$

The magnetic boundary conditions are that the normal component of magnetic induction and tangential component of magnetic field are continuous across the boundary,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial x}, \quad \frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial y}, \quad M_z + \frac{\partial\phi}{\partial z} = \frac{\partial\psi}{\partial z}, \quad z = \pm\frac{1}{2}, \tag{33}$$

where ψ is the magnetic potential outside of fluid, and it satisfies

$$\nabla^2\psi = 0, \quad |z| \geq \tfrac{1}{2}. \tag{34}$$

In order to match the domain of Chebyshev pseudospectral-QZ method, we reset the present domain from $[-\frac{1}{2},\frac{1}{2}]$ to $[-1,1]$ with coordinate transformation of z to $2z$ in equations (26), (28), (30), and (31) and in the above boundary conditions. We perform the standard normal mode analysis and look for the solution of variables w, M_z, ϕ, θ in the form

$$\begin{bmatrix} w \\ M_z \\ \phi \\ \theta \end{bmatrix} = \begin{Bmatrix} w(z) \\ M_z(z) \\ \phi(z) \\ \theta(z) \end{Bmatrix} \times \exp[(k_x x + k_y y)i + \sigma t]. \tag{35}$$

On substituting (35) into equations (26), (28), (30), and (31), in the new domain, we obtain:

$$\begin{aligned}\sigma\frac{1}{\mathrm{Pr}}(4D^2 - k^2)w(z) &= \{16D^4 - 8k^2D^2 + k^4\}w(z) \\ &\quad + \Big\{-2\chi_2M_5(M_1z - 2M_3)D^3 - 4\chi_2M_1M_5D^2 \\ &\quad + \frac{\chi_2M_5}{2}(M_1z - 2M_3)k^2D + \chi_2M_1M_5k^2\Big\}M_z(z) \\ &\quad + \Big\{-4M_5(M_1z - 2M_3)D^4 - 8M_1M_5D^3 \\ &\quad + M_5(M_1z - 2(2+\chi_2)M_3)k^2D^2 + 2M_1M_5k^2D \\ &\quad + \frac{(1+\chi_2)}{2}M_3M_5k^4\Big\}\phi(z) - k^2\ \mathrm{Rg}\ \theta(z),\end{aligned} \tag{36}$$

$$\sigma M_z(z) = -\frac{\chi M_4}{\chi_2(1+\chi)}w(z) - \frac{1}{\tau}M_z(z) + 2\frac{\chi}{\chi_2\tau}D\phi(z) - \frac{\chi M_4}{\chi_2\tau}\theta(z), \tag{37}$$

$$\begin{aligned}\sigma(4D^2 - k^2)\phi(z) &= \Big\{\frac{\chi M_4}{1+\chi}zD^2 + 2\chi_2D^2 + \frac{2\chi M_4}{1+\chi}D \\ &\quad - \frac{\chi M_4k^2}{4(1+\chi)}z - \frac{\chi_2k^2}{2}\Big\}w(z) + \Big\{\frac{\chi_2M_1\ \mathrm{Rg}\ z^2}{8\xi_1(1+\chi)} \\ &\quad + \frac{\chi_2(\chi_2-1)M_3\mathrm{Rg}\ z}{4\xi_1(1+\chi)} - \frac{\chi_2^2M_3M_5}{2\xi_1}\Big\}DM_z(z) \\ &\quad + \Big\{\Big(\frac{M_1\mathrm{Rg}z^2}{4\xi_1(1+\chi)} + \frac{(\chi_2-1)M_3\ \mathrm{Rg}\ z}{2\xi_1(1+\chi)} \\ &\quad - \frac{\chi_2M_3M_5}{\xi_1} - \frac{4(1+\chi)}{\tau}\Big)D^2 + (1+\chi_2) \\ &\quad \times\Big(\frac{\chi_2M_3M_5}{4\xi_1} + \frac{1}{\tau}\Big)k^2 \\ &\quad + \frac{(1+\chi_2)M_3\mathrm{Rg}k^2}{8\xi_1(1+\chi)}z\Big\}\phi(z) + 2\frac{\chi M_4}{\tau}D\theta(z),\end{aligned} \tag{38}$$

$$\sigma\frac{(1+\chi)M_2}{\chi}\left(z - \frac{2}{M_4}\right)D\phi(z) + \sigma\theta(z) = \Big\{(1 - M_2) + \frac{1}{2}M_2M_4z\Big\}w(z) + \{4D^2 - k^2\}\theta(z), \tag{39}$$

where derivative $D = \partial/\partial z$ and wave number $k^2 = k_x^2 + k_y^2$. The boundary conditions for velocity w and temperature θ are

$$w=\frac{\partial w}{\partial z}=\theta=0 \quad \text{at } z=\pm 1. \tag{40}$$

Following Refs. [8,20], the boundary conditions for magnetic field are

$$M_z+2\frac{\partial \phi}{\partial z}+k\phi=0 \quad \text{at } z=1, \tag{41}$$

$$M_z+2\frac{\partial \phi}{\partial z}-k\phi=0 \quad \text{at } z=-1. \tag{42}$$

III. CHEBYSHEV PSEUDOSPECTRAL METHOD AND THE EIGENVALUE PROBLEM

We now solve Eqs. (36)–(39) with boundary conditions (40)–(42) by pseudospectral (collocation) method [21]. We remark that the main motivations for the use of spectral methods in numerical calculations stems from the approximation properties of orthogonal polynomial expressions. Moreover, the approximation errors in spectral methods decrease much faster than algebraical methods.

We expand the unknown functions in Chebyshev polynomials $T_n(x)=\cos(n \arccos x)$ on $[-1,1]$ as

$$f(x)=\sum_{n=0}^{L} a_n T_n(x), \tag{43}$$

where

$$a_n=\frac{2}{c_n\pi}\int_{-1}^{1}\frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx. \tag{44}$$

Here $C_0=2,\ c_i=1\ (i\geqslant 1)$.

We now replace integration in (44) with numerical discrete integral on the collocation points,

$$a_n=\frac{2}{c_nL}\sum_{j=0}^{L}\frac{1}{c_j}f(x_j)T_n(x_j), \tag{45}$$

where x_j are called collocation points in $[-1,1]$. The most optimal choice, for most problems, of these collocation points are Gauss-Lobatto collocation points $x_j=-\cos(j\pi/L)$, $j=0,1,2,\ldots$. Equation (43) can then be expressed as

$$f(x)=\sum_{j=0}^{L} g_j(x)f(x_j), \tag{46}$$

where the interpolating polynomial is given by

$$g_j(x)=\frac{2}{Lc_j}\sum_{n=0}^{L}\frac{1}{c_n}T_n(x_j)T_n(x). \tag{47}$$

After introducing the Gauss-Lobatto integration, the c_j in (47) will be $c_0=c_L=2$, $c_j=1$, for $1\leqslant j\leqslant L$. Equation (46) implies that the derivative of $f(x)$ can be represented by derivatives of the interpolating polynomials $g_i(x)$ given by (47). The differentiation matrix $\mathbf{D}$ may be introduced as

$$\mathbf{D}_{i,j}=g_j'(x_i)=\frac{2}{Lc_j}\sum_{k=0}^{L}\frac{1}{c_k}T_k(x_j)T_k'(x_i). \tag{48}$$

For the second, third, and fourth derivatives, there are relationships of the type $\mathbf{D}^2=\mathbf{D}\mathbf{D}^2$, $\mathbf{D}^{n+1}=\mathbf{D}\mathbf{D}^n$. The derivative values on the Gauss-Lobatto collocation points can be expressed by their values on the same points,

$$\mathbf{F}^{(n)}=\mathbf{D}^n\mathbf{F}. \tag{49}$$

Components of matrix $\mathbf{D}$ are dependent on the order L of Chebyshev polynomials. After discretization, equations of the generalized eigenvalue problem usually have the form

$$\sigma\begin{bmatrix}\mathbf{B}_{1,1} & \mathbf{B}_{1,2}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{1,1} & \mathbf{A}_{1,2}\\ \mathbf{BC}_{2,1} & \mathbf{BC}_{2,2}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix}. \tag{50}$$

The equation $\mathbf{BC}_{2,1}\mathbf{X}_1+\mathbf{BC}_{2,2}\mathbf{X}_2=\mathbf{0}$ represents boundary conditions. Variable vector $\mathbf{X}_2$ are the boundary points and the outermost internal points for Chebyshev pseudospectral method. If matrix $\mathbf{BC}_{2,2}$ is not singular, the variable vector $\mathbf{X}_2$ can be condensed and Eq. (50) becomes

$$\sigma[\mathbf{B}_{1,1}-\mathbf{B}_{1,2}\mathbf{BC}_{2,2}^{-1}\mathbf{BC}_{2,1}]\mathbf{X}_1=[\mathbf{A}_{1,1}-\mathbf{A}_{1,2}\mathbf{BC}_{2,2}^{-1}\mathbf{BC}_{2,1}]\mathbf{X}_1. \tag{51}$$

We checked this algorithm and applied it to the treatment of the Bénard problem as an example. We found excellent matching with the results of previous authors. We used the QZ algorithm, subroutine DGGEV in LAPACK, to solve the general eigenvalue problem. We tried for order $N+1\geqslant 16$, and found that almost all leading eigenvalues in this case, have identical values at least up to five digits.

On returning to the general eigenvalue equations (36)–(39), we note that these can be discretized by Chebyshev pseudospectral method in the new domain $[-1,1]$ and written in the matrix form as

$$\sigma\mathbf{B}\mathbf{X}=\mathbf{A}\mathbf{X}, \tag{52}$$

where $\mathbf{A},\mathbf{B}$ are nonsymmetric complex matrices, given as

$$\mathbf{B}=\begin{bmatrix}\frac{1}{\mathrm{Pr}}(4\mathbf{D}^2-k^2\mathbf{I}) & 0 & 0 & 0\\ 0 & \mathbf{I} & 0 & 0\\ 0 & 0 & 4\mathbf{D}^2-k^2\mathbf{I} & 0\\ 0 & 0 & \frac{(1+\chi)}{\chi}\left(\mathbf{Z}-\frac{2}{M_4}\mathbf{I}\right)M_2\mathbf{D} & \mathbf{I}\end{bmatrix}, \tag{53}$$

TABLE I. Physical properties of ferrofluid (T_a=298 K), from Rosensweig [2].

	Magnetic staturation Ms (A/m)	Density ρ (kg/m3)	Viscosity η [kg/(m s)]	Thermal conductivity Kt [kg m/(K s3)]	Thermal expansion Coeff. α 1/K	Specific heat C [J/(m3 K)]	Heat capacity $Cp,H=C/\rho$ [J/(kg K)]	Prandti number Pr	Volume fraction φ	Votex viscosity ξ [kg/(m s)]	Brownian relax time τm s
Hydrocarbon I	15900	1050	0.003	0.15	9.0×10^{-4}	1.72×10^{6}	1633.33	32.7	0.036	1.60×10^{-4}	1.15×10^{-6}
Hydrocarbon II	31800	1250	0.006	0.19	8.6×10^{-4}	1.84×10^{6}	1472.00	46.5	0.071	6.42×10^{-4}	2.29×10^{-6}
Ester I	15900	1150	0.014	0.31	8.1×10^{-4}	3.72×10^{6}	3238.26	146.2	0.036	7.49×10^{-4}	5.35×10^{-6}
Ester II	31800	1300	0.030	0.31	8.1×10^{-4}	3.72×10^{6}	2864.62	277.2	0.071	3.21×10^{-3}	1.15×10^{-5}
Ester III	47700	1400	0.035	0.31	8.1×10^{-4}	3.72×10^{6}	2660.00	300.3	0.107	5.61×10^{-3}	1.34×10^{-5}
Water I	15900	1180	0.007	0.59	5.2×10^{-4}	4.18×10^{6}	3545.76	42.1	0.036	3.74×10^{-4}	2.67×10^{-6}
Water II	31800	1380	0.010	0.59	5.0×10^{-4}	4.18×10^{6}	3031.88	51.4	0.071	1.07×10^{-3}	3.82×10^{-6}

$$\mathbf{A} = \begin{bmatrix} 16\,\mathbf{D}^4 - 8k^2\mathbf{D}^2 + k^4\mathbf{I} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} & -k^2\,\mathrm{Rg}\,\mathbf{I} \\ -\dfrac{\chi M_4}{\chi_2(1+\chi)}\mathbf{I} & -\dfrac{1}{\tau}\mathbf{I} & \dfrac{2\chi}{\chi_2\tau}\mathbf{D} & -\dfrac{\chi M_4}{\chi_2\tau}\mathbf{I} \\ \mathbf{A}_{3,1} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \dfrac{2\chi M_4}{\tau}\mathbf{D} \\ \dfrac{1}{2}M_2M_4\mathbf{Z} + (1-M_2)\mathbf{I} & 0 & 0 & 4\,\mathbf{D}^2 - k^2\mathbf{I} \end{bmatrix}, \tag{54}$$

and

$$\mathbf{A}_{1,2} = -2\chi_2 M_5(M_1\mathbf{Z} - 2M_3)\mathbf{D}^3 - 4\chi_2 M_1 M_5\mathbf{D}^2 + \frac{\chi_2 M_5}{2}(M_1\mathbf{Z} - 2M_3)k^2\mathbf{D} + \chi_2 M_1 M_5 k^2\mathbf{I},$$

$$\mathbf{A}_{1,3} = -4M_5(M_1\mathbf{Z} - 2M_3)\mathbf{D}^4 - 8M_1M_5\mathbf{D}^3 + M_5[M_1\mathbf{Z} - 2(2 + \chi_2)M_3]k^2\mathbf{D}^2 + 2M_1M_5k^2\mathbf{D} + \frac{(1+\chi_2)}{2}M_3M_5k^4\mathbf{I},$$

$$\mathbf{A}_{3,1} = \frac{\chi M_4}{1+\chi}\mathbf{Z}\mathbf{D}^2 + 2\chi_2\mathbf{D}^2 + \frac{2\chi M_4}{1+\chi}\mathbf{D} - \frac{\chi M_4 k^2}{4(1+\chi)}\mathbf{Z} - \frac{\chi_2 k^2}{2}\mathbf{I},$$

$$\mathbf{A}_{3,2} = \frac{\chi_2 M_1\,\mathrm{Rg}}{8\xi_1(1+\chi)}\mathbf{Z}^2\mathbf{D} + \frac{\chi_2(\chi_2-1)M_3\,\mathrm{Rg}}{4\xi_1(1+\chi)}\mathbf{Z}\mathbf{D} - \frac{\chi_2^2 M_3 M_5}{2\xi_1}\mathbf{D},$$

$$\mathbf{A}_{3,3} = \frac{M_1\,\mathrm{Rg}}{4\xi_1(1+\chi)}\mathbf{Z}^2\mathbf{D}^2 + \frac{(\chi_2-1)M_3\,\mathrm{Rg}}{2\xi_1(1+\chi)}\mathbf{Z}\mathbf{D}^2 - \frac{\chi_2 M_3 M_5}{\xi_1}\mathbf{D}^2 - \frac{4(1+\chi)}{\tau}\mathbf{D}^2 + \frac{(1+\chi_2)M_3\,\mathrm{Rg}\,k^2}{8\xi_1(1+\chi)}\mathbf{Z} + (1+\chi_2) \times\left(\frac{\chi_2 M_3 M_5}{4\xi_1} + \frac{1}{\tau}\right)k^2\mathbf{I}.$$

Here $k^2 = k_x^2 + k_y^2$, matrix $\mathbf{D}$ is differentiation matrix introduced in (48), $\mathbf{I}$ is an identity matrix, and Z is the diagonal coordinate matrix, and $\mathbf{Z}_{i,i} = -\cos(i\pi/N)$, $i=0,\ldots,N$. The vector $\mathbf{X} = [\mathbf{w}, \mathbf{M}_z, \boldsymbol{\phi}, \boldsymbol{\theta}]^T$ is variable in w, M_z, ϕ, θ on the Gauss-Lobatto collocation points. Boundary conditions in discrete form are

$$w_0 = w_N = \sum_{j=1}^{N-1} D_{0,j}w_j = \sum_{j=1}^{N-1} D_{N,j}w_j = \theta_0 = \theta_N = 0,$$
$$M_{z0} - k\phi_0 + 2\sum_{j=0}^{N} D_{0,j}\phi_j = M_z N + k\phi_N + 2\sum_{j=0}^{N} D_{N,j}\phi_j = 0. \tag{55}$$

Equation (52) together with boundary conditions (55) may by arranged in the form (50), which may be condensed in the form of Eq. (51).

We now follow the procedure and algorithms, as described in Ref. [22]. For a given β, wave number k and H_a with other physical parameters, we employ the QZ algorithm subroutine **DGGEV** of LAPACK library, to solve the generalized eigenvalue equations (52). We then follow the algo-

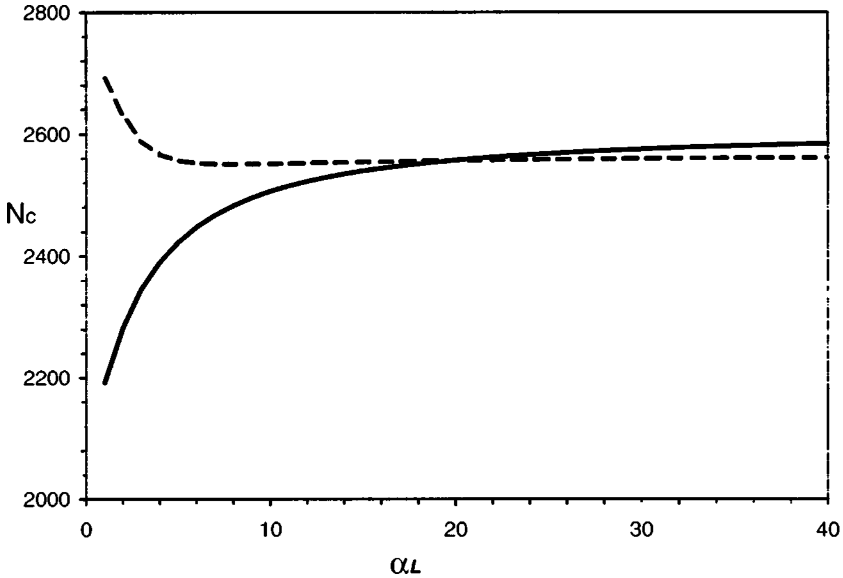

FIG. 3. Plots of the critical magnetic Rayleigh number N_c (gravity free) of a waterbased ferrofluid against α_L, solid line represents (ξ,τ) consideration and dotted line represents without (ξ,τ). The thickness of fluid layer, in each case, is $d=2$ mm.

rithm for the determining neutral stability curves. Using QZ algorithm, we find leading eigenvalue $s=s_R+i\omega$ for corresponding wave number k. Adjusting β by secant method, we get the temperature gradient β when the real part s_R of the leading eigenvalue $s=s_R+i\omega$ is zero. From the neutral stability curves (β,k), the critical temperature gradient β with critical wave number k_c can be defined as

$$\beta_c = \min_k \beta(P_r, H_a, \ldots). \tag{56}$$

The minimization of Eq. (56) is carried out by means of the golden section search routine. We point out that if the imaginary part ω of leading eigenvalue $s=s_R+i\omega$ happens to be zero and at the same time, if its real part s_R approaches zero, the stability is stationary. Otherwise, the stability is oscillatory.

IV. NUMERICAL RESULTS AND DISCUSSION

We investigate real ferrofluids in our calculations with fixed magnetite particles ($M_d=4.46\times10^5$ A/m) and diameters ($d=10$ nm). Table I lists some physical properties of ferrofluids which we have taken from Ref. [2]. On calculating the concentration $\varphi=M_s/M_d$, we find they are all below 10% and thus could be considered as dilute ferrofluids. Moreover the relation functions (11) and (15) are then applicable for different parameters. In the following calculation we have estimated vortex viscosity ξ by the formula $\xi=\frac{3}{2}\eta\varphi$ and Brownian relaxation time $\tau_m=3V\eta/k_BT$. We have also set $M_2=0$ since its value is negligible.

Figure 1 shows the plots of χ, the tangent susceptibility, χ_2, chord susceptibility, and the Langevin function $L(\alpha_L)$ against the Langevin parameter α_L. It is apparent that χ, χ_2, and $L(\alpha_L)$ cannot be treated as constants. Figure 2 shows the variation of the K_m, the pyromagnetic coefficient, against α_L. Here, it is again difficult to assume it to be constant. We also point out that we carried out calculations for the parameter M_3 in Finlayson [8], and found that, for real ferrofluids, it varied between $1\leqslant M_3\leqslant 1.5$ for a wide range of value of α_L. Finally we remark that we had considered all seven kinds of ferrofluids in our calculations but, because of the similarity in results in several cases, we have presented results for a few selected ones.

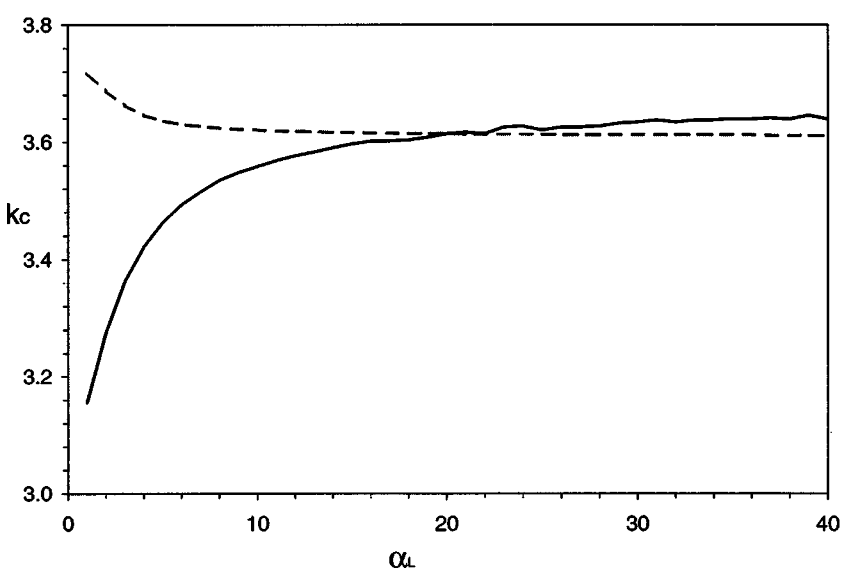

FIG. 4. Plot of critical wave number k_c against α_L for the same fluid and in similar cases as in Fig. 3.

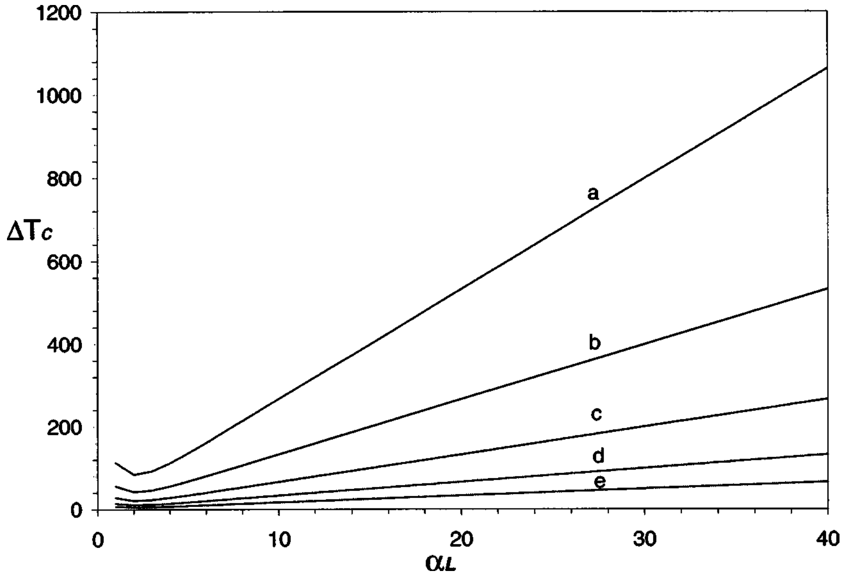

FIG. 5. Variation of critical temperature ΔT_c (gravity free) for a waterbased ferrofluid with different thickness of the fluid layer d, (a) d=1 mm, (b) d=2 mm, (c) d=4 mm, (d) d=8 mm, trace (e) d=16 mm without (ξ,τ) consideration.

We begin our discussion with the situation when gravity is absent that is when convection is driven by magnetic forces only, as it turned out to be quite striking in comparison to the previous results [8,20]. In the following we identify, for comparison purposes, by (ξ,τ) the vortex viscosity and relaxation time combination, representing the theory of particle rotation used in this paper.

Figure 3 shows the plot of N_C, the critical magnetic Rayleigh number against α_L for both cases, with (ξ,τ) present and (ξ,τ) absent. We did not find much variation in values for different thicknesses d and thus have presented results for d=2 mm. For higher value of α_L, we find our results nearly agreeing with those of Finlayson [8] and Stiles and Kagan [20] as expected, but the results for the lower values of α_L are new and interesting. In the case of particle rotation theory, when (ξ,τ) are taken into account, we find that the instability sets in at lower values of N_C. As a matter of fact, it is found to be most unstable at lowest values of α_L. As α_L increases so does N_C and, eventually above α_L=20, it takes on the value near 2570. On the other hand, in the absence of (ξ,τ), we find a contrasting situation. Now at the lower values of α_L, the fluid is more stable; the instability setting in at higher values of N_C. However, as α_L increases, N_C now decreases and eventually around α_L=30 it again takes the value near 2560. The exact values of N_C depend upon the nature of the ferrofluid considered. We point out that in this case our results complement the results of Ref. [20] for a complete spectrum of values of α_L. We thus conclude that the consideration of (ξ,τ), in the present case is to accelerate the instability.

A somewhat parallel situation occurs when critical wave number k_c is plotted against α_L. Figure 4 shows the plots of both situations. We note that, in the presence of (ξ,τ), the wave number increases as α_L increases and around α_L=15 it stabilizes to a value of 3.60. On the other hand, when the effect of (ξ,τ) is absent, the value of k_C starts higher at 3.714 at α_L=1 and drops to 3.610 around α_L=20. The critical temperature ΔT_c however not only varies with the variation of

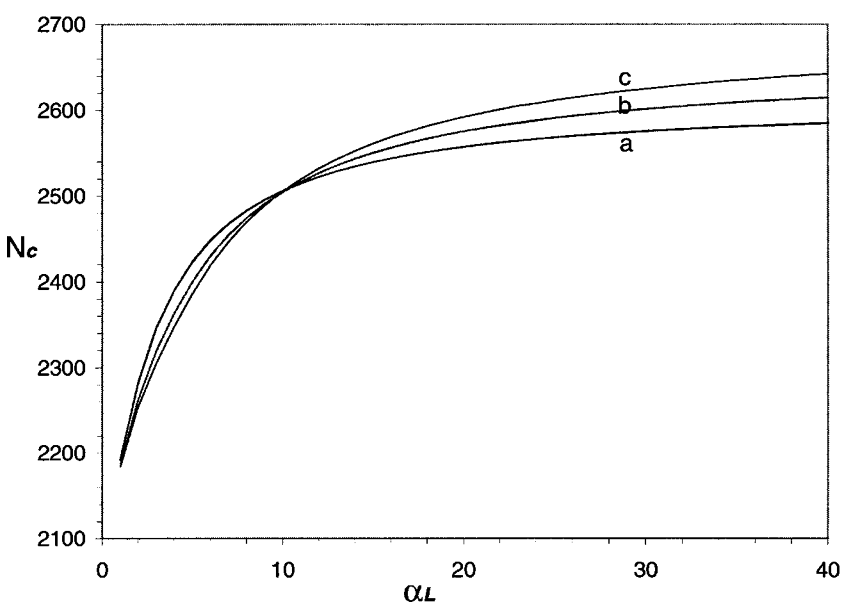

FIG. 6. Critical N_c for different base ferrofluids (gravity free), with (ξ,τ), d=2 mm, (a) Ester I, Water I, Hydrocarbon I; (b) Ester II, Water II, and Hydrocarbon II; (c) Ester III.

TABLE II. Critical Rg_c, N_c… With gravity for Ester I, with or without (ξ, τ).

		With (ξ, τ)				Without (ξ, τ)			
d(mm)	α_L	Rg_c	k_c	N_c	ΔT_c	Rg_c	k_c	N_c	ΔT_c
1	0	1707.8	3.115	0.000	218.020	1707.8	3.116	0.000	218.020
	1	676.1	3.146	1326.209	86.316	734.9	3.481	1566.785	93.819
	2	553.5	3.232	1550.023	70.667	588.5	3.509	1751.698	75.123
	4	711.2	3.314	1415.900	90.788	730.5	3.436	1493.905	93.255
	5	822.6	3.322	1283.754	105.010	836.8	3.398	1328.735	106.830
	7	1018.5	3.312	1034.796	130.020	1023.6	3.335	1045.300	130.680
	10	1230.8	3.284	749.397	157.130	1224.5	3.271	741.716	156.320
	15	1438.6	3.245	457.878	183.660	1419.1	3.209	445.526	181.160
	20	1548.7	3.222	299.147	197.720	1521.3	3.175	288.651	194.220
1.5	0	1707.7	3.115	0.000	64.598	1707.8	3.116	0.000	64.598
	1	1135.5	3.136	738.881	42.953	1201.9	3.313	827.809	45.462
	2	987.9	3.196	975.136	37.367	1034.0	3.361	1068.329	39.113
	4	1179.8	3.240	769.772	44.628	1196.3	3.289	791.448	45.251
	5	1292.8	3.236	626.387	48.902	1298.5	3.252	631.915	49.116
	7	1452.3	3.220	415.634	54.936	1442.3	3.203	409.916	54.558
	10	1578.3	3.201	243.405	59.700	1555.1	3.166	236.313	58.823
	15	1665.5	3.185	121.220	62.997	1632.7	3.140	116.495	61.758
	20	1700.7	3.178	71.256	64.329	1663.9	3.130	68.205	62.939
2	0	1707.8	3.118	0.000	27.252	1707.8	3.116	0.000	27.252
	1	1425.4	3.130	368.409	22.746	1470.5	3.210	392.094	23.466
	2	1312.7	3.170	544.797	20.948	1348.7	3.250	575.126	21.524
	4	1464.6	3.193	375.333	23.372	1466.4	3.199	376.278	23.402
	5	1538.6	3.190	280.733	24.554	1529.3	3.177	277.339	24.404
	7	1626.8	3.182	165.006	25.960	1604.3	3.151	160.478	25.602
	10	1684.5	3.176	87.730	26.880	1653.3	3.134	84.510	26.384
	15	1719.3	3.170	40.875	27.436	1682.6	3.125	39.147	26.850
	20	1732.5	3.171	23.397	27.646	1693.4	3.121	22.353	27.022
4	0	1707.8	3.114	0.000	3.407	1707.8	3.116	0.000	3.407
	1	1688.6	3.125	32.311	3.368	1688.4	3.124	32.304	3.368
	2	1681.3	3.138	55.860	3.354	1673.6	3.129	55.348	3.338
	4	1711.6	3.153	32.038	3.414	1687.9	3.123	31.157	3.367
	5	1722.0	3.156	21.976	3.435	1694.1	3.121	21.271	3.379
	7	1733.1	3.158	11.705	3.457	1700.5	3.119	11.269	3.392
	10	1740.2	3.163	5.852	3.471	1704.2	3.118	5.612	3.399
	15	1744.9	3.164	2.631	3.481	1706.1	3.117	2.516	3.403
	20	1747.0	3.166	1.487	3.485	1706.8	3.117	1.419	3.405
8	0	1707.8	3.123	0.000	0.426	1707.8	3.116	0.000	0.426
	1	1712.2	3.125	2.076	0.427	1706.5	3.117	2.063	0.426
	2	1720.7	3.135	3.657	0.429	1705.5	3.118	3.592	0.425
	4	1733.1	3.150	2.053	0.432	1706.5	3.117	1.990	0.426
	5	1736.5	3.153	1.397	0.433	1706.9	3.117	1.350	0.426
	7	1740.7	3.158	0.738	0.434	1707.3	3.117	0.710	0.426
	10	1743.9	3.161	0.367	0.435	1707.5	3.116	0.352	0.426
	15	1746.6	3.161	0.165	0.435	1707.7	3.116	0.158	0.426
	20	1747.9	3.166	0.093	0.436	1707.7	3.116	0.089	0.426

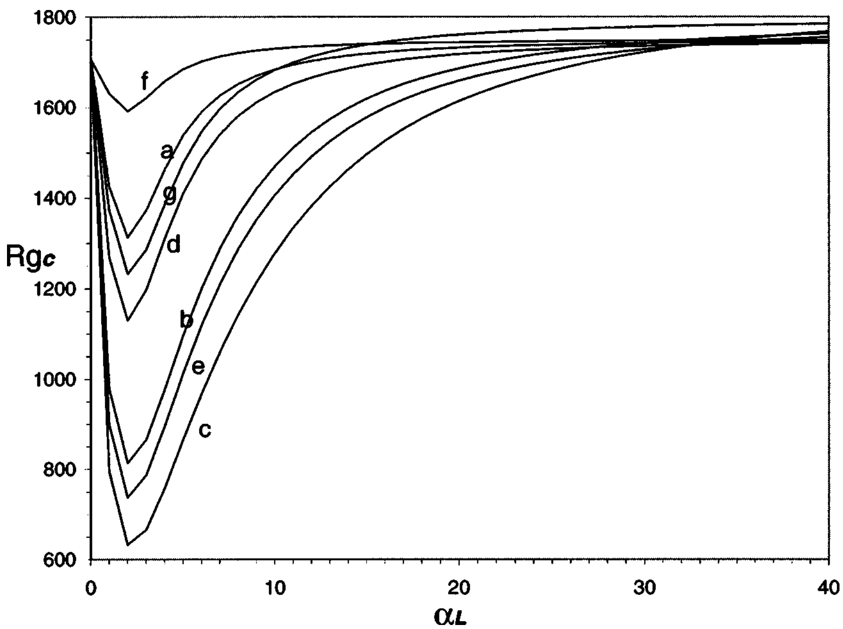

FIG. 7. Critical Rg_c for different ferrofluids, with gravity and (ξ,τ), $d=2$ mm, (a) Ester I, (b) Ester II, (c) Ester III, (d) Water I, (e) Water II, (f) Hydrocarbon I, (g) Hydrocarbon II.

α_L (it increases with the increase of α_L) but it also varies with the thickness d.

Figure 5 shows the variation of ΔT_c with α_L for different thicknesses of fluid layer d. As expected as the thickness d increases, ΔT_c decreases, implying a lower value of ΔT to kick in the convection at higher values of d. we did not find the effect of consideration of or neglect of (ξ,τ), in this case.

Figure 6 shows the effect of varying (ξ,τ) combinations. This is accomplished by plotting the variation of N_C against α_L for different ferrofluids and thus for different (ξ,τ) combinations. We found results, for different base ferrofluids, to be identical and, therefore, these are grouped together. It can be seen that the ferrofluid with Ester I base is the most unstable amongst them because it has lowest vortex viscosity ξ. We carried out calculation for the critical wave number k_c for different ferrofluids. Similar to the behavior of N_C, we find that as ξ increases so does the critical wave number k_c and k_c increases with increase of α_L.

For the discussion of convection when both gravitational and magnetic fields are applid simultaneously, we give Table II. This table displays both the cases when (ξ,τ) is taken into consideration and when (ξ,τ) effect is absent. As can be seen from the table, our first observation, in this case, is about the importance of the thickness of the layer. In the absence of gravitational effect, we recall that we did not find significant effect of the variation of the thicknesses d. In the present case, we note that critical Rayleigh number Rg_c has a very low value at $\alpha_L=2$ and it increase as α_L increases. This pattern continues as the thickness of the layer is increased. We note that the magnetic effect dominates when the thicknesses d is a small. After $d=4$ mm or so, the buoyancy force, however, takes over much of the magnetic force and the critical Rg takes on higher values, even at small α_L. An exactly opposite pattern is observed for the critical N_c. It takes on maximum value when $d=1$ mm and when α_L is also small, and then continues decreasing as α_L increases. It also decreases as d increases.

Finlayson [8] has proposed a formula,

$$\frac{\mathrm{Rg}_c}{\mathrm{Rg}_{0c}}+\frac{N_c}{N_{0c}}=1, \tag{57}$$

in which $Rg_{0c}=1708$ and $N_{0c}=2568.5$, to indicate the tight coupling of the convection mechanism between buoyancy and magnetic forces. We recall that Nield [23] had earlier proposed a similar formula to predict the tight coupling between buoyancy and surface tension forces. In the present case we note that, in the absence of (ξ,τ) consideration, the above formula holds within a couple of percent. In the presence of (ξ,τ) consideration, however, we find that the formula did not quite apply particularly at lower values of α_L and at small thickness d.

A glance at the values of the wave number shows that their values in both cases, with (ξ,τ) or without (ξ,τ) consideration, are almost the same. With regard to the critical temperature gradient to start the convection, we note that it increases as α_L increases and that there is very small difference whether (ξ,τ) is considered or not. In comparison to the gravity free situation we find that ΔT_c is much higher in gravity free case as compared to the case with gravity. In comparison to the viscous fluid case we note that ΔT_c is always lower in the magnetic fluid case. We looked for, but did not find, the oscillatory behaviors in all cases. Our final Fig. 7 represents the effect of varying ξ, the vortex viscosity on Rg_c. From this figure, we note that ferrofluid with Ester III base is most unstable with reference to the Rg, as compared to other ferrofluids.

We close this section with the following remarks. In this paper, we have for the first time considered the effect of theory of particle rotation and the vortex viscosity for the onset of thermal convection in a thin layer. We found that the effect of consideration of (ξ,τ) is most effective in the thin layer and at lower values of α_L and that the results are surprisingly different there. We have noted that the buoyancy forces become dominant once the thickness is increased. We have not only considered the effect of consideration of τ and ξ but have extended, in the absence of (ξ, τ) the results of Finlayson [8] and Stiles and Kagan [20] to a complete range of magnetic field. It would have been more advisable, in the numerical calculation, to leave τ and ξ as arbitrary variables, but in view of the fact that the theory used is applicable to dilute ferrofluids, our choices for τ and ξ are justifiable and satisfactory.

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